Director Walter's method is practicable only in schools where the teacher has a limited number of small classes.

Neuphilologisches Centralblatt furnishes many articles equal in interest and importance to those I have mentioned. The progressive teacher will be amply repaid for devoting some time to the careful reading of this journal.

RADICAL AND CONSERVATIVE ELEMENTS IN THE TEACHING OF MATHEMATICS[1]

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Valuable as is the mental training possible when mathematics is wisely taught, we question its right to a place on the curriculum if the subject-matter has no value in itself. We do not wish to put this on a mere commercial basis, but on the basis of a larger and a fuller life—that is, a greater efficiency, not for selfish ends, but for the advancement of whatever is most worth while. It is possible that the mental training obtained in mathematics is more rigorous, but not essentially different from that obtained in other subjects, so that if the pupil has no real use for mathematics, he would better take something else.

If, however, we consider the future of our pupils, at least the commercial value of mathematics is evident, not only to boys who may enter mechanical pursuits, but also to future teachers and those who may enter business houses. We believe that mathematics should be taught, not from the point of view of the specialist, but from the point of view of the child and as a tool.

It has been suggested that what is of value in any course, is the point of view, not the mass of details, which must of necessity be forgotten. The unity and value of the algebra course center in the presentation of the equation as an instrument in solving problems, while the practical value of the geometry lies in a knowledge of its theorems and general principles. Mental training must follow of necessity if our work is done as it should be, but our work is a failure if the essential content of the subject is not made the center about

[1] Read at the conference of the Department of Mathematics.

which all details are grouped, and if the pupil cannot clearly perceive the underlying unity and gain knowledge and power.

If we consider the application of these ideas to the teaching of algebra and geometry, some interesting points suggest themselves. One is a change in the order of topics. This is allowable, even when a definite text is used, for the right of the teacher to think for himself and plan his own course is a divine right. We believe in textbooks however. Our pupils need to learn to study independently, and the best text available should be chosen with this object. We question the success of all attempts to do away with a text.

If the pupils are to be made to feel the importance of the equation in such a way that they will never forget it, the order in most texts seems to serve the purpose but poorly. This order is something like the following: the four fundamental operations with integers, factoring, fractions, simple equations with one unknown, simple simultaneous equations, exponents, surds, quadratic equations, and so on. Occasionally some work with integral equations is given with the four fundamental operations, and fractional equations are given with the work on fractions. Problems are invariably given in connection with the different kinds of equations. If the book is to be of value later as a reference-book—and this is essential—this is a good order; but it is an objectionable order in teaching the subject, as it appears to treat the equation as only one topic among many.

The point of view may be brought out nicely by introducing work in equations at every possible opportunity. Not only, for example, may transposition be taught with addition and subtraction, but also one method of elimination in simultaneous equations. If the pupil's knowledge of numerical fractions is wisely used, fractional equations with numerical denominators and the remaining methods of elimination may be taught with multiplication. As soon as the pupil can write the square of a binomial, he may be taught the reverse process, which is usually postponed until factoring. He may then be given easy affected quadratic equations. Even simultaneous equations, one of which is of first degree and one of second degree, may be introduced quite early. Harder fractional equations may be sprinkled along with factoring and fractions. Verification of roots will utilize work in fractions and radicals.

It will be seen that everything that it is worth while for the pupil to know has a place in this scheme. As we have said, fractions and radicals are necessary in verification of equations. Square root of numbers has a place in the solution of quadratic equations with irrational roots. Such roots should not only be given in the usual form, as $\frac{1}{2} \pm \frac{1}{2}\sqrt{3}$, but should be obtained approximately to three or four decimal places. The square root of polynomials is not directly essential for equations, but is essential for a thorough comprehension of square root of numbers.

There are a few topics usually included in the textbooks whose utility we seriously question. Highest common divisor by division is an example. We wonder, also, if cube root and binomial theorem have a place in secondary work. It certainly is not good pedagogy to introduce the pupil to a miscellaneous collection of topics from advanced subjects just as fast as he is capable of solving elementary problems in these subjects, if the problems have no bearing on the rest of the work.

I have given my advanced algebra class a diagram similar to that given below, for the purpose of showing the relation of the various topics studied to the equation.

Problems form another interesting topic. Not only are many of the problems given in the algebras of little value in themselves, but the classification usually adopted seems to me to serve no purpose whatsoever. What inspiring or valuable idea is a pupil to get from a collection of problems on miscellaneous subjects, grouped together simply because they all give rise to quadratic equations, or to simultaneous equations? If one follows the order of the text closely, such a collection resolves itself into a set of puzzles, to the solution of which the pupil is left largely to his own resources. If the subject-matter of the problem involves mathematical relations with which he is familiar, he solves the problem. If this is not the case, he gives it up. This is of necessity so because if the order of the text is followed, no systematic attempt can be made to enable pupils to understand the relations involved in any one particular subject. Each set of problems contains a few on that subject. A few are solved today, and a few next month, and the majority of the class knows no more about these questions at the end of the year than at the beginning. In other

Kinds of Equations	Methods of Solution	Verification	Applications	Graphs
Linear				
Quadratic				
Higher				
Radical				

words, the authors of textbooks introduce problems simply as illustrations of certain kinds of equations, not with the idea that the teacher is to go systematically back of the equation to the mathematical relations involved in certain subjects; and clearly to bring out these relations and to train the pupil to translate them into algebraic language. This has not been considered the province of the algebra teacher.

Dr. Tompkins' remark, "Push your subject to its utmost, and you do all the correlating necessary," seems to me to define the legitimate limits of correlation, and to apply both to the teacher of physics and chemistry and to the teacher of algebra and geometry. If a problem involves a technical knowledge of another subject that cannot be given by a few words of explanation, that problem and the algebra necessary to solve it properly belong to that other subject, and the teacher has no right to object to a review of the necessary

mathematics. On the other hand, it seems to me that a training in certain applications of algebra is a vital part of the algebra work. Among the most important of such applications are problems involving time, rate, and distance, problems involving mensuration or arithmetical relations with which the pupil is familiar. The manipulation of formulæ from physics and arithmetic, so as to solve for the various letters involved, seems to me to come under the same head.

If, as has been suggested, the pupil is early introduced to the various kinds of equations, consideration of any one kind of problem is a comparatively easy matter. Take, for example, the questions involving time, rate, and distance. After the relations have been thoroughly discussed, the different kinds of problems can be taken up, the easiest first, without regard to the nature of the equations produced. There are cases of one body meeting another, or of one body overtaking another, and cases involving the resultant of two forces. All these cases must be thoroughly discussed before more complicated problems are given.

At this point I wish to pay my respects to the laboratory method as opposed to the oral recitation. Dr. G. Stanley Hall, in an address last year, spoke of the short circuit from the ear to the tongue as more natural than the long circuit from the eye to the hand. Are we in danger of neglecting the short circuit, and thus not only depriving the pupil of a natural and legitimate means of acquiring knowledge—that is, by the general discussion—but also of giving insufficient training in oral expression? The laboratory method is often very fruitful for a time, especially if a class is a little discouraged. It is well, then, to set the pupils at work on a set of easy problems. But the class has a right to the oral recitation with the general interchange of ideas and criticisms afforded.

Another topic of interest is the relation of the concrete to the abstract. I think that perhaps the real question, is not which of these two elements should come first, but what kind of concrete questions should come first and what kind last. Concrete problems which are taken from the pupil, and which he may reasonably be supposed to solve, properly lead up to something new or abstract; but I cannot make myself believe that it is ever wise to assign tasks that the pupil has absolutely no means of solving. Many tasks

given in this way seem to me to belong at the end as applications of the abstract.

One of the most serious problems in algebra-teaching has long been how to give algebra a meaning to a class of beginners.

There have been attempts made to solve this problem by the introduction of graphs and elementary physics experiments. Because these things give to us concrete illustrations of abstract ideas, we have assumed that they do to the child. As a matter of fact, they are as surely outside of his experience as is the mathematics, however concrete they may be in themselves. It may be a simpler matter to enlarge this experience so as to take in the things desired for illustration, than it is to enlarge it to take in the algebra. These illustrations may embody information useful and desirable in themselves, yet the fact remains that before they can be effectually used the new must be thoroughly assimilated by means of what is already in the mind, and the round-about connection established between the child's life and the illustration, and then between the illustration and the subject in hand. Even when this connection is worked out to the satisfaction of the teacher, the child fails utterly to grasp it. This may be due in part to hurried work, but more often to the fact that children are not capable of appreciating a round-about or long argument. The result to the child is a mass of disconnected, unintelligible details not dominated by any controlling idea. After a short time nothing remains.

It has been suggested that if arithmetic were used judiciously it might be made to solve the problem before us. Let the pupil verify every equation. Let him verify the various equations obtained in the solution of the given equation, so as to appreciate the difference between changing the value and changing the form. Let him pass from the concrete to the general form of the same problem, and, having solved a literal equation, use the answer as a formula to obtain the roots of various numerical equations of which this is a type. Let him make up problems to fit certain equations. Let him evaluate both the problem and the answer in cases where simplification of more or less complex expressions is desired. Let algebraic fractions be approached through arithmetical fractions. It may be that by the end of the year the general nature of algebra will have dawned

on his immature mind. At least we have established a direct, and not a round-about, connection between algebra and the child's life. Indeed, I find the puzzle instinct stronger in my pupils than any interest in the mathematical relations of physical phenomena. Perhaps by appealing to the pleasure that a healthy mind takes in its own exercise, and using what information the pupil already possesses, I can develop a broader interest in the world about.

Just here I wish to express my appreciation of the work done by Professor C. E. Comstock, of Bradley Polytechnic Institute, Peoria, Ill. It seems to me the most suggestive and practical of all that we have heard in the last two years. Many of the ideas suggested above were received from him.

The most important thing in geometry is the subject-matter, and if we are to teach geometry, that should be the thing of prime importance. This does not mean that it is necessary for the pupil to retain text-proofs longer than the few recitations that they are under discussion. The important question is: Can the pupils apply theorems and general principles in inventing proofs and solving problems? To bring out these principles, it is frequently useful to alter the order of the text, where this can be done to advantage without destroying the sequence used. In Book III, for example, one important theorem is: "If two triangles have their angles equal, their sides are proportional." A number of theorems usually given at the end of Book III are directly dependent on this one. The proofs are not long. If these are given as originals immediately after the theorem quoted, and if this work is then followed by a number of easy exercises in which it is required to obtain a proportion by means of one pair of similar triangles, the classroom gets the notion pretty well fixed.

We insist also on the training in formal reasoning. It is because this phase of the subject is entirely new that geometry is always so hard at first. At the beginning, therefore, the work should be especially planned to meet this need. Since the author of any text must of necessity determine a definite order and strictly preserve the sequence, and since this almost invariably results in putting some of the hardest proofs at the beginning, it seems to me not only legitimate, but necessary, to say to a class: "Assume for the time being that you have proved such and such a theorem, and prove this or

that other theorem;" and this because training and formal reasoning must be given at the outset.

Examine from this standpoint the introductory theorems in any of the textbooks. In Phillips & Fisher, for example, the first eleven theorems involve a large number of geometrical principles, a variety of kinds of proofs, and some of the longest arguments in the first book. Is it any wonder that the pupil becomes discouraged before he conquers them?

The desired result is obtained much more quickly and easily by some exercises like the following. After the theorem about vertical angles has been proved, easy problems can be given to illustrate algebraic methods. For example, if (in the figure of two lines with a third line intersecting each of them, the eight angles being suitably numbered), $\angle 4 + \angle 6 = 2\ rt\ \angle s$, why does $\angle 6 = \angle 3$? why does $\angle 7 = \angle 3$? and so on. The pupil can then learn the theorem that makes alternate interior angles of parallel lines equal, and, assuming that it has been proved, prove various theorems, corollaries, and exercise dependent directly upon it, and so on. It is thus possible to grade work carefully, to give much drill in theorem-making and translating, and to group together proofs of the same nature. It will not be found necessary to assume the proofs to more than two or three theorems.

I am frequently told that by skipping about in this way the pupil is given no opportunity to appreciate the sequence. I do not wish to be misunderstood. In assuming proofs the assumption is freely admitted and no interdependent proofs are allowed. But while the pupil soon comes to appreciate the relation between two theorems, the appreciation of the sequence of the whole is a thing of slow and later growth and is best brought out in reviews.

What has been said about the introduction of graphs and physics experiments into algebra applies equally well to some exercises introduced into geometry. The question is not whether a particular experiment is permissible in itself. Anything is permissible which serves a purpose toward a clearly defined and legitimate end, and is presented so as to train the power to solve problems. But the introduction of exercises the sole object of which is to assist in formation of clear images, may be questioned, if the exercise is of no value in

itself, and if that image is not made to function toward some practical end. We are not in the entertaining business, and even clear images may be an incumbrance.

All of us have long been influenced by a wrong psychology. The mind is not a collection of faculties, but a unit and as a unit must be trained. Its highest function, as far as we know, is the solution of the problems that are continually confronting us. If our training does not make for greater efficiency in this respect, it is worse than useless. We have perhaps spent too much time on details that are of interest only to the specialist, and have often failed to give the point of view that inspires, or the training in making application of knowledge that really trains. But in trying to remedy our shortcomings let us not repeat our mistakes in another form. Let us not read ourselves into the children, nor imagine that what is clear to us is clear to them, or that what is in our lives is of necessity in their lives also. Let us be sure that we have a correct estimate of values, and see that we give as directly and as efficiently as possible that which is of most value. Let us not expect too much of the children, but in true breadth of spirit train in all things that make for perfect manhood.

DISCUSSION

PROFESSOR YOUNG, of the University of Chicago: May not the very real difficulties in applying mathematics in physics be remedied to a considerable extent by the simultaneous teaching of algebra, geometry, and physics throughout the high-school course, each subject not necessarily every week or every month, but at least every year? This seems to me the next great step forward to be taken in the teaching of mathematics in this country. (In Chicago a mere redistribution of the hours now given to physics and to mathematics would suffice, without altering the total number of hours given to either.) As a nation we are conservative in educational matters. In England the "Perry movement," about five years old in its present form, has worked a real revolution in the teaching of geometry. In France and Germany, algebra, geometry, and physics are taught side by side, and, as a rule, the teachers of physics there find no such marked deficiency as ours report in ability to apply the mathematics previously learned.

PROFESSOR E. H. MOORE: We are all conservative-radicals. There is perhaps a mistaken idea that we think we are emphasizing novelties. The result to be reached is that students understand not only the theory, but also that *it fits the phenomena of nature*, from which it arises by the mediation of the thinker,

and to which it has continuous application. To judge from my own experience in college work center of gravity experiments and problems would be of value in secondary work.

AN INQUIRER: Has Professor Young worked out a definite plan for the simultaneous teaching of algebra, geometry, and physics?

MR. YOUNG: I have no special plan of my own, but the method has long been in vogue in Germany and France; the official curricula are quite detailed and constitute the working plans.

AN INQUIRER: Are the foreign curricula mentioned by Professor Young accessible in English?

MR. YOUNG: I know of none but my little book on *The Teaching of Mathematics in Prussia* (Longmans).[1]

MISS SYKES: By teaching algebra and geometry together, alternating between them, energies are scattered too much.

MR. MOORE: Algebra, geometry, and physics should always be in the field of vision. The line of sight may be directed to some one in particular, but without losing sight of the rest.

MR. BRESLICH, of the University High School (recently of Bradley Polytechnic Institute, Peoria): The plan was tried at Bradley, but failed. Students forgot their geometry while attending to algebra, and conversely. Confusion was also caused by the fact that some students passed in one subject, but not in the other. The work was hampered by lack of a suitable text.

MISS SYKES: Comstock's *Algebra*, mimeographed, is valuable for definite concrete problems.

MR. BRESLICH: I think Comstock's book will be published this year. The mimeographed copies are sold by Professor Comstock, Peoria, for 75 cents, while they last.

MR. STOUT, of Howe School, Lima, Ind.; MR. COBB, of Lewis Institute; MR. FIELD, of the Academy of Northwestern University; MR. LENNE, of John Marshall High School, Chicago, and MR. WICKES, of the University High School, gave reports of the simultaneous teaching of algebra and geometry with, on the whole, encouraging results.

MR. YOUNG, in answer to the question, "What effect would our elective system have on this simultaneous teaching? Is there an elective system in Germany?" said: There is no elective system in Germany, but the plan is feasible

[1] The official curricula are: *Lehrpläne und Lehraufgaben für die höhere Schulen in Preussen* (Berlin: Cotta; pp. 76); curricula of 1901. *Plan d'études et programmes d'enseignement dans les lycées et collèges de garçons* (Paris: Delalain; pp. xl+208); curricula of 1902. The textbooks usually treat the subjects separately, leaving, as do the official instructions, the order of development of the work allotted to each year to the discretion of the teacher. Mehler's *Elemente der Mathematik* is much used in Prussia. Jules Tannery's *Notions de Mathématiques* (Paris: Delagrave, 1903; pp. x+352) is very suggestive, especially along the line of interrelation of algebra and geometry, and early use of idea of derivatives. There are good English books on German higher schools in general by Russell (Longmans) and Bolton (Appleton).

under our system. The experiment is certainly hampered by fixing definite days for algebra and others for geometry, and by giving credit for one without the other. The work should be treated and credited as a unit—a year's work in mathematics.

PROFESSOR MYERS, of the School of Education of the University of Chicago: We should perceive clearly that improvement is possible and strive to make it. However much algebra and geometry may be mixed up, we must realize that the effect on the students' mind should be unified. Trigonometry is pre-eminently a high-school subject. Personally, I introduce some surveying and mechanics in algebra. It is to be deplored that there is no literature on the subject, and that consequently each teacher is left to thresh out his material as best he may. One of the pressing needs of the day is the preparation of suitable texts.

MR. MOORE: However various our expressions, we surely are united in purpose. Let us then agree: (1) To press for consideration the merits of the concurrent teaching of algebra and geometry, as one subject, *mathematics*; (2) even with the curriculum as at present, to introduce into first-year algebra many problems having origin in observational geometry, and into geometry many problems requiring algebra in their solution or of algebraic origin (e. g., construction to scale of numerical expressions); (3) to report results in next year's conference.

WHAT AMOUNT OF MECHANICS IS IT DESIRABLE TO INTRODUCE INTO A FIRST-YEAR COURSE IN PHYSICS, AND IN WHAT POSITION SHOULD IT COME?[1]

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The *order* in which he presents the various parts of his subject is a matter given careful attention by the thoughtful teacher of physics. The varied arrangement of our textbooks shows that this question of best order of topics has been attacked from many standpoints, and that the true solution will not be reached by examining one set of data.

The question before us takes up one phase of the larger question just indicated. In its consideration questions similar to the following present themselves: Is there danger of giving too much time to mechanics at the expense of more important topics? Is mechanics, on the other hand, of such fundamental importance that there is no

[1] Read at the conference of the Department of Physics.